Topology and its Applications 19 (1985) 1-11
North-Holland

A NOTE ON HULL OPERATORS IN $(\mathscr{E}, \mathscr{M})$ CATEGORIES

Harriet Lazowick LORD

Université Laval, and Champlain Regional College, St. Lawrence Campus, Canada

Received 26 June 1981
Revised 5 August 1982 and 20 June 1983

We characterize those operators on the objects of $\mathscr{C}$—a 'nice' concrete category—which correspond to strong factorization structures on $\mathscr{C}$ that are not necessarily additive, and show that these operators behave like convex hulls (as opposed to topological closure in the additive case.) Moreover, those subobjects of X which are invariant under an operator Q are precisely the $\mathscr{M}$-subobjects of X, where $(\mathscr{E}, \mathscr{M})$ is the unique strong factorization corresponding to Q.

AMS Subj. Class. (1980): Primary 18A32; Secondary: 46A05, 46A55, 55B30.

concrete category
factorization structure
hull operator
closure operator
limit operator
affine hull

Introduction

The relationship between factorization structures and closure operators has been studied by several authors. For example, in [8] Nakagawa characterizes $\mathscr{M}$-subobjects for strong additive factorization structures $(\mathscr{E}, \mathscr{M})$ on *TOP* by means of a closure operator which has also been studied by Herrlich [4].

An example of such a closure operator is topological closure. The corresponding factorization structure is $(\mathscr{E}_{\mathscr{C}}, \mathscr{M}_{\mathscr{C}})$, where $\mathscr{M}_{\mathscr{C}}$ is the family of closed embeddings, and $\mathscr{E}_{\mathscr{C}}$ is the family of maps with dense images. Since the union of two closed subspaces is a closed subspace, $(\mathscr{E}_{\mathscr{C}}, \mathscr{M}_{\mathscr{C}})$ is additive in the sense that the union of two $\mathscr{M}_{\mathscr{C}}$-subobjects is an $\mathscr{M}_{\mathscr{C}}$-subobject.

In this note, we look at factorization structures on 'nice' concrete categories which are strong (by strong we mean $\mathscr{M} \subseteq$ embeddings) but not necessarily additive. An example of such a structure can be obtained as follows:

The notion of closed convex hull in Euclidean spaces can be extended to *FUNSP*, a category of continuous function spaces on topological spaces [see example 3 following Definition 1.1.]. This convex hull operator induces a factorization structure $(\mathscr{E}_{\mathscr{C}}, \mathscr{M}_{\mathscr{C}})$ on *FUNSP*, where $\mathscr{M}_{\mathscr{C}}$ is the family of embeddings with convex images, and $\mathscr{E}_{\mathscr{C}}$ is the family of maps whose codomains are the convex hulls of their images. Since the union of two convex sets is not necessarily convex, $(\mathscr{E}_{\mathscr{C}}, \mathscr{M}_{\mathscr{C}})$ is not additive.

It is the preceding example which has motivated our choice of the terminology 'hull operator' instead of closure operator, since closure operators are additive. Each hull operator, denoted Q, induces a strong factorization structure $(\mathscr{E}_Q, \mathscr{M}_Q)$, where $\mathscr{M}_Q$ is the class of embeddings whose images are their hulls, and $\mathscr{E}_Q$ is the class of morphisms the hull of whose images are their codomains.

For each strong factorization structure $(\mathscr{E}, \mathscr{M})$ we obtain a hull subobject operator $\mathscr{R}_{\mathscr{M}}$, where $\mathscr{R}_{\mathscr{M}}$ assigns to each object the class of all its $\mathscr{M}$-subobjects. In *HAUS*, for example, the (epi, extremal mono) factorization yields the closed subspaces. In *FUNSP*, the extremal mono-subobjects are the function space affine subspaces. (In Euclidean spaces, these are the closed affine subspaces.) This hull operator in *FUNSP* is denoted H-aff S, where $S \subseteq X$ and H is a space of continuous real-valued functions on S. This is the example which motivates our definition of affine hull.

Each hull subobject operator induces, in turn, a hull operator. The correspondence between hull operators and hull subobject operators is analogous to the situation in topological spaces in which for each space X we can define either the closure of a subspace S to be the union of S with the set of all its limit points, or equivalently, we can assign to X the class of its closed subspaces, and define the closure of S to be the intersection of all closed subspaces of X that contain S.

These results are summarized in Theorem 3.1, where we show that there is a 1-1 correspondence between the classes of hull operators, hull subobject operators, and strong factorization structures.

In Section 1 we give preliminary results and define embeddings, strong factorization structures, and affine hulls. Hull operators and hull subobject operators are presented in Section 2. Section 3 contains our main result—Theorem 3.1. In Section 4, we briefly look at possible modifications of $(\mathscr{E}, \mathscr{M})$; the first is the case where $\mathscr{M} \not\subseteq$ embeddings, and the second is the case $\mathscr{E} \subseteq Epi$.

1. Preliminaries

$\mathscr{C}$ is a complete, well-powered concrete category with forgetful functor $\mathscr{T}: \mathscr{C} \to$ *SET* that preserves monomorphisms.

Definition 1.1. [5] A monomorphism $f: X \to Y$ in $\mathscr{C}$ is an *embedding* iff for each morphism $g: Z \to Y$ in $\mathscr{C}$ and each function $h: \mathscr{T}(Z) \to \mathscr{T}(X)$ such that $\mathscr{T}(g) = \mathscr{T}(f)h$ there exists a unique morphism $\bar{h}: Z \to X$ such that $g = f\bar{h}$. We denote the class of embeddings by $\mathscr{M}_0$.

Examples. 1) In *TOP*, $\mathscr{M}_0$ is the class of topological embeddings. ($\mathscr{M}_0$ is the class of extremal monomorphisms.)

2) In *HAUS*, $\mathscr{M}_0$ is the class of topological embeddings. ($\mathscr{M}_0$ contains all extremal monomorphisms.)

3) Let *FUNSP* be the category whose objects are all pairs (X, H), where X is a topological space and H is a linear subspace of $C(X)$, the continuous real-valued functions on X, which contains the constants; and $f: (X, H) \to (Y, K)$ is a morphism iff $f: X \to Y$ is continuous and $Kf \subseteq H$. f is an embedding iff $f: X \to Y$ is a topological embedding and $Kf = H$. [7]

4) If $\mathscr{C}$ is an algebraic category (e.g. *GRP*, *SGRP* (semi-groups with semigroup homomorphisms)) the embeddings are the monomorphisms. [5]

Those properties of embeddings that will be used in this note are given in the following proposition:

Proposition 1.2. [5] *$\mathscr{M}_0$ is the class of embeddings.*

1) *$\mathscr{M}_0$ is closed under the formation of intersections, compositions, pullbacks, and products;*

2) *Every extremal monomorphism in $\mathscr{C}$ is an embedding;*

3) *If fg is an embedding, then g is an embedding.*

Remark. Since $\mathscr{T}$ preserves monomorphisms, f is an embedding iff it is a $\mathscr{T}$-initial lifting of a monomorphism. This coincides with the definition of embedding in *Nel* [9].

Definition 1.3. Let $f: S \to X$ be an embedding. Then (S, f) will be called an $\mathscr{M}_0$-*subobject* of X.

Definition 1.4. ([6] and [5]) Let $\mathscr{E}$ and $\mathscr{M}$ be classes of morphisms of $\mathscr{C}$. $(\mathscr{E}, \mathscr{M})$ is a *factorization structure* for $\mathscr{C}$ iff

1) $\mathscr{E}$ is closed under compositions;

2) $\mathscr{M}$ is closed under compositions;

3) $\mathscr{E} \cap \mathscr{M}$ contains all isomorphisms;

4) Each morphism f in $\mathscr{C}$ is uniquely factorizable; i.e. $f = me$, where $m \in \mathscr{M}$, $e \in \mathscr{E}$, and this factorization is unique in the sense that if $f = m'e'$, where $m' \in \mathscr{M}$, $e' \in \mathscr{E}$, then there is an isomorphism h such that $he' = e$ and $mh = m'$.

If $(\mathscr{E}, \mathscr{M})$ is a factorization structure for $\mathscr{C}$, $\mathscr{C}$ will be called an $(\mathscr{E}, \mathscr{M})$ *category*.

Definition 1.5. If $(\mathscr{E}, \mathscr{M})$ is a factorization structure for $\mathscr{C}$, and each $m \in \mathscr{M}$ is an embedding, $(\mathscr{E}, \mathscr{M})$ will be called a *strong factorization structure.* (We point out that our definition differs from that in [8].)

For the remainder of this article, we make the following additional assumption for the category $\mathscr{C}$: *There exists a class of epimorphisms $\mathscr{E}_0$ such that $\mathscr{C}$ is an $(\mathscr{E}_0, \mathscr{M}_0)$ category.*

Definition 1.6. Let $f: X \to Y$ be a morphism in $\mathscr{C}$, with $f = me$, $m \in \mathscr{M}_0$, $e \in \mathscr{E}_0$, where

$m: C \to Y$. The $\mathscr{M}_0$-subobject of Y, (C, m), is the *image* of f and will be denoted by either $(f(X), m)$ or, more simply, $f(X)$.

The following proposition shows that if (S, i) is an $\mathscr{M}_0$-subobject of X, $f(S)$ is well-defined.

Proposition 1.7. *Let $f: X \to Y$ be a morphism in $\mathscr{C}$, and (S, i) be an $\mathscr{M}_0$-subobject of X. $fi: S \to Y$. If $f = me$ and $fi = m_1 e_1$ are the $(\mathscr{E}_0, \mathscr{M}_0)$ factorizations of f and fi respectively, then $m_1 e_1 = mm_2 e_2$, where $ei = m_2 e_2$ is the $(\mathscr{E}_0, \mathscr{M}_0)$ factorization of ei.*

Proof. Consider the following commutative diagrams:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "e"'] & & Y \\ & f(X) \arrow[ur, "m"'] & \end{tikzcd} \qquad \begin{tikzcd} S \arrow[rr, "fi"] \arrow[dr, "e_1"'] & & Y \\ & f(S) \arrow[ur, "m_1"'] & \end{tikzcd} \qquad \begin{tikzcd} S \arrow[rr, "ei"] \arrow[dr, "e_2"'] & & f(X) \\ & C \arrow[ur, "m_2"'] & \end{tikzcd}$$

$fi = mei = mm_2 e_2 = m_1 e_1$.

Images also behave well with respect to compositions; i.e. $(gf)(X) = g(f(X))$.

Proposition 1.8. *Let $f: X \to Y$ and $g: Y \to Z$ be morphisms in $\mathscr{C}$. Consider the following $(\mathscr{E}_0, \mathscr{M}_0)$ factorizations:*

$$\begin{tikzcd} X \arrow[rr, "gf"] \arrow[dr, "e_1"'] & & Z \\ & gf(X) \arrow[ur, "m_1"'] & \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[rr, "g"] \arrow[dr, "e_2"'] & & Z \\ & g(Y) \arrow[ur, "m_2"'] & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "e_3"'] & & Y \\ & f(X) \arrow[ur, "m_3"'] & \end{tikzcd}$$

$$\begin{tikzcd} f(X) \arrow[rr, "gm_3"] \arrow[dr, "e_4"'] & & Z \\ & C \arrow[ur, "m_4"'] & \end{tikzcd}$$

Then $m_1 e_1 = m_4 e_4 e_3$.

Proof. $m_1 e_1 = gf = gm_3 e_3 = m_4 e_4 e_3$.

We now introduce the notion of affine hull. The remainder of this section deals with properties of this hull.

Definition 1.9. Let (S, i_S) be an $\mathscr{M}_0$-subobject of X. The *X-affine hull of* (S, i_S), denoted $(X\text{-aff}(S, i_S), j_S)$ is defined as follows:

$(X\text{-aff}(X, i_S), j_S) = \bigcap\{(U, i_U) : (U, i_U)$ is an $\mathscr{M}_0$-subobject of X, i_U is an extremal monomorphism, and there exists $\alpha_U \in \mathscr{M}_0$ such that (S, α_U) is an $\mathscr{M}_0$-subobject of U and $i_S = i_U \alpha_U\}$. (i.e., $(X\text{-aff}(S, i_S), j_S)$ is the intersection of all extremal mono-subobjects of X that 'contain' S.)

Examples. 1) In *TOP*, $(X\text{-aff}(S, i_S), j_S) = (S, i_S)$, i_S is an inclusion map.

2) In *HAUS*, $(X\text{-aff}(S, i_S), j_S) = (cl_X S, j_S)$, i_S, j_S are inclusion maps.

3) In *FUNSP*, $((X, H)\text{-aff}((S, H|S), i_S), j_S) = ((H\text{-aff } S, H|H\text{-aff } S), j_S)$, where $H\text{-aff } S = \{x \in X : h(S) \Rightarrow h(x) = 0 \text{ for all } h \in H\}$, and i_S, j_S are inclusions.

4) In *GRP*, $(X\text{-aff}(S, i_S), j_S) = (S, i_S)$, i_S is an inclusion.

Proposition 1.10. *There exists a unique embedding $i: S \to X$-aff (S, i_S) such that $j_S i = i_S$. Furthermore, i is an epimorphism.*

Proof. That i exists and is unique follows from the definition of intersection.

To show that i is an epimorphism, let $i = me$, where m is an extremal monomorphism, e an epimorphism. $e: S \to Z$, $m: Z \to X$-aff (S, i_S). Then there exists $d: X$-aff $(S, i_S) \to Z$ so that the following diagram commutes:

$$\begin{tikzcd}
S \arrow[r, "i"] \arrow[d, "e"'] & X\text{-aff}(S, i_S) \arrow[d, "\mathrm{id}"] \arrow[dl, dashed, "d"'] \\
Z \arrow[r, "m"'] & X\text{-aff}(S, i_S)
\end{tikzcd}$$

and $j_S md = j_S$. Thus $m = d^{-1}$ is an isomorphism and i is an epimorphism.

Definition 1.11. Let (S, i_S) be an $\mathscr{M}_0$-subobject of X and $\alpha: X \to Y$ a morphism in $\mathscr{C}$. Then by $\alpha|X$ we mean the morphism $\alpha i_S: S \to Y$.

Proposition 1.12. *Let $\alpha, \beta: X \to Y$ be such that $\alpha|S = \beta|S$, where (S, i_S) is an $\mathscr{M}_0$-subobject of X. Then $\alpha|X\text{-aff}(S, i_S) = \beta|X\text{-aff}(S, i_S)$; i.e. $\alpha i_S = \beta i_S \Rightarrow \alpha j_S = \beta j_S$.*

Proof. The proof is immediate since $i: S \to X\text{aff}(S, i_S)$ is an epimorphism.

Proposition 1.13. *Let $f: X \to Y$ be a morphism in $\mathscr{C}$ and $f = me$ its $(\mathscr{E}_0, \mathscr{M}_0)$ factorization. f is an epimorphism if and only if $Y = Y$-aff $(f(X), m)$.*

Proof. Assume f is an epimorphism. m has a factorization $m = ji$, where $(Y\text{-aff}(f(X), m), j)$ is the Y-affine hull of $f(X)$, and i is the epimorphism described in Proposition 1.10. Since j is an extremal monomorphism, j must be an isomorphism.

Now assume that j is an isomorphism. Then $f = me = jie$ is the composition of epimorphisms.

Proposition 1.14. *Assume that each extremal monomorphism in $\mathscr{C}$ is regular. Let (S, i_S) be an $\mathscr{M}_0$-subobject of X, (B, k) an $\mathscr{M}_0$-subobject of X such that there exists an embedding j' such that $(X\text{-aff}\,(S, i_S), j')$ is an $\mathscr{M}_0$-subobject of B with $j = kj'$, and (B, k) has the property that whenever $\alpha|S = \beta|S$, then $\alpha|B = \beta|B$ for all morphisms α, β with domain X and a common codomain. Then $B = X\text{-aff}\,(S, i_S)$.*

Proof. j' is an extremal monomorphism since j is. Thus j and j' are regular. Therefore there exist morphisms α, β, and γ such that the following diagram commutes:

$$\begin{tikzcd} X\text{-aff } S \arrow[rr, "j"] \arrow[dr, shift right, "j'"'] & & X \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & Y \\ & B \arrow[ul, dashed, shift right, "\gamma"'] \arrow[ur, "k"'] & & \end{tikzcd}$$

$j\gamma = k$ and thus γ is an embedding. $k = kj'\gamma$ and $j\gamma j' = j$, so $j'\gamma = \text{id}$ and $\gamma j' = \text{id}$. Thus $j' = \gamma^{-1}$.

That the assumption that all extremal monomorphisms in $\mathscr{C}$ are regular cannot be deleted from the above proposition is shown as follows:

Assume that (S, i_S) is an extremal mono-subobject of X that is not regular. Let (B, j) be the intersection of all those regular subobjects (T, i_T) of X which satisfy the property that there exists an embedding $k_S: S \to T$ with $i_S = i_T k_S$. Then (B, j) satisfies the hypothesis of Proposition 1.14, and is regular. Thus $(S, i_S) \neq (B, j)$.

2. Hull operators

In this section we define and give examples of hull operators. We then define hull subobject operators and show that these are equivalent to hull operators.

Definition 2.1. A *hull operator* on $\mathscr{C}$ is an operator Q that assigns to each $\mathscr{M}_0$-subobject (S, i_S) of every object X an $\mathscr{M}_0$-subobject of X called the *$X - Q$ hull* of (S, i_S), denoted $(X - Q \textit{ hull } (S, i_S), j_S))$, which satisfies the following conditions:

1) There exists $i': S \to X - Q\ \text{hull}(S, i_S) \in \mathscr{M}_0$ such that $j_S i' = i_S$;

2) If $i_A: A \to B$, $i_B: B \to X \in \mathscr{M}_0$, and $i'_A: A \to X$ with $i'_A = i_B i_A$, then there exists $j' \in \mathscr{M}_0$ such that $j_A = j_B j'$, where $j_B: X - Q\ \text{hull}(B, i_B) \to X$, $j_A: X - Q\ \text{hull}(A, i'_A) \to X$, and $j': X - Q\ \text{hull}(A, i'_A) \to X - Q\ \text{hull}(B, i_B)$;

3) If $i_A: A \to Y$, $i_Y: Y \to X \in \mathscr{M}_0$, $i_Y = j_Y$, then $j_A = i_Y j'_A$, where $j_A: X - Q\ \text{hull}(A, i_Y i_A) \to X$, j'_A: $Y - Q\ \text{hull}(A, i_A) \to Y$;

4) If $f: X \to Y$ is a morphism, $i_Z: Z \to Y \in \mathscr{M}_0$, $j_Z: Y - Q$ hull$(Z, i_Z) \to Y$, $i_Z = j_Z$, and the following square is a pullback

$$\begin{tikzcd} E \arrow[r, "p_2"] \arrow[d, "p_1"'] & X \arrow[d, "f"] \\ Z \arrow[r, "i_Z"'] & Y \end{tikzcd}$$

then $p_2 = j_E$, where $j_E: X - Q$ hull$(E, p_2) \to X$;

5) If $(T, i_T) = (X - Q\,\text{hull}(S, i_S), j_S)$, then $i_T = j_T = j_S$, where $j_T: X - Q$ hull $(T, i_T) \to X$.

Examples. 1) *Affine hull* (Definition 1.9) is a hull operator on $\mathscr{C}$.

2) *Idempotent limit-operators.* ([4] and [8]). In *TOP* (and in *HAUS*), let $X - Q$ hull$(S, i_S) = (l_X S, j_S)$, where l_X is an idempotent limit-operator, j_S is the inclusion. For example, $l_X S = \text{cl}_X\, S$.

3) The *trivial hull operator.* $X - Q$ hull$(S, i_S) = (S, i_S)$.

4) In *FUNSP* [7]

a) *H-convex hull.* If $(X, H) \in ob$ *FUNSP* and $S \subseteq X$, H-conv $S = \{x \in X: h(x) \leq \sup h(S)$ for all $h \in H\}$.

b) *H-affine hull.* H-aff $S = \{x \in X$: $h(S) = 0$ implies $h(x) = 0$ for all $h \in H\}$.

(If X is a compact convex subspace of a locally convex space E and $H = A(X)$, the continuous real affine functions on X, then H-aff S is the closure of the intersection of the geometric affine hull of S in E with X.)

5) *Isolator.* ([2] and [10]) Let S be a sub(semi)group of a (semi)group G. The isolator (or closure) of S in G is defined as

$$I(S) = \bigcup_{k=1}^{\infty} I_k(S),$$

where $I_1(S)$ is the sub(semi)group of G generated by $\{x \in G$: $x^n \in S$ for some positive integer $n\}$, $I_k(S) = I_1(I_{k-1}(S))$. If G is abelian, $I(S) = I_1(S) = \{x \in G$: $x^n \in S$ for some positive integer $n\}$.

Definition 2.2. A *hull subobject operator* on $\mathscr{C}$ is an operator R that assigns to each object X a class of $\mathscr{M}_0$-subobjects of X, denoted $R(X)$, which satisfies the following conditions:

1) $R(X)$ is closed under intersections;

2) If $f: X \to Y$ is a morphism, $(S, i_S) \in R(Y)$, and the following square is a pullback

$$\begin{tikzcd} E \arrow[r, "p_2"] \arrow[d, "p_1"'] & X \arrow[d, "f"] \\ S \arrow[r, "i_S"'] & Y \end{tikzcd}$$

then $(E, p_2) \in R(X)$;

3) If $(Y, i_Y) \in R(X)$, then $(S, i_S) \in R(Y)$ iff$(S, i_Y i_S) \in R(X)$.
4) $(X, \mathrm{id}) \in R(X)$.

Lemma 2.3. *Each hull subobject operator on $\mathscr{C}$ induces a hull operator on $\mathscr{C}$.*

Proof. Let R be a hull subobject operator on $\mathscr{C}$. We define a hull operator Q_R as follows: $(X - Q_R\,\mathrm{hull}(A, i_A), j_A) = \bigcap\{(S, i_S)\colon (S, i_S) \in R(X),\ (A, i'_A)$ is an $\mathscr{M}_0$-subobject of S with $i_A = i_S i'_A.\}$. Q_R obviously satisfies conditions 1, 2, 4, and 5 of Definition 2.1. To show condition 3, let $(Y, i_Y) = (X - Q\,\mathrm{hull}(Y, i_Y), j_Y)$. $(Y - Q_R\,\mathrm{hull}(A, i_A), j'_A) = \bigcap\{(S, i_S) \in R(Y)\colon (A, i'_A)$ an $\mathscr{M}_0$-subobject of S with $i_A = i_S i'_A\}$. Since $(S, i_S) \in R(Y)$ iff$(S, i_Y i_S) \in R(X)$, $X - Q_R\,\mathrm{hull}(A, i_Y i_A) = Y - Q_R\,\mathrm{hull}(A, i_A)$ and $j_A = i_Y j'_A$.

Lemma 2.4. *Each hull operator on $\mathscr{C}$* induces a hull subobject operator on $\mathscr{C}$.

Proof. Let Q be a hull operator on $\mathscr{C}$. Define a hull subobject operator as follows: $R_Q(X) = \{(S, i_S)\colon (X - Q\,\mathrm{hull}(S, i_S), j_S) = (S, i_S)\}$.

That $R_Q(X)$ is closed under intersections follows from condition 4 of Definition 2.1. Conditions 2 and 4 of Definition 2.2 are obvious. For condition 3, let $(Y. i_Y) \in R_Q(X)$. Then $(Y, i_Y) = (X - Q\,\mathrm{hull}(Y, i_Y), j_Y)$. $R_Q(Y) = \{(S, i_S)\colon (Y - Q\,\mathrm{hull}(S, i_S), j'_S) = (S, i_S)\}$. Thus $i_S = j'_S$. If $j_S\colon X - Q\,\mathrm{hull}(S, i_Y i_S) \to X$, then $j_S = i_Y i_S$, and $(S, i_Y i_S) \in R_Q(X)$. If $(S, i_Y i_S) \in R_Q(X)$, then $j_S = i_Y i_S$. Since $i_Y \in \mathscr{M}_0$, $i_Y j'_S = i_Y i_S$ implies $i_S = j'_S$. Therefore $(Y - Q\,\mathrm{hull}(S, i_S), j'_S) = (S, i_S)$, and $(S, i_S) \in R_Q(Y)$.

Proposition 2.5. *$R_{Q_R} = R$ and $Q_{R_Q} = Q$.*

Proof. Let R be a hull subobject operator on $\mathscr{C}$, and Q_R the induced hull operator. Denote R_{Q_R} by R'. Then $R'(X) = \{(S, i_S)\colon (X - Q_R\,\mathrm{hull}(S, i_S), j_S) = (S, i_S)\}$. Thus if $(S, i_S) \in R'(X)$, then $(S, i_S) = \bigcap\{(T, i_T)\colon (T, i_T) \in R(X),\ (S, i'_S)$ is an $\mathscr{M}_0$-subobject of T with $i_S = i_T i'_S\}$. Since $R(X)$ is closed under intersections, $(S, i_S) \in R(X)$. On the other hand, if $(S, i_S) \in R(X)$, then $(S, i_S) = (X - Q_R\,\mathrm{hull}(S, i_S), j_S)$ and thus $(S, i_S) \in R'(X)$. Therefore we have $R(X) = R'(X)$ and so $R = R' = R_{Q_R}$.

Let Q be a hull operator on $\mathscr{C}$, and let Q' denote Q_{R_Q}. Then $(X - Q'\,\mathrm{hull}(A, i_A), j_A) = \bigcap\{(S, i_S)\colon (S, i_S) \in R_Q(X),\ (A, i'_A)$ is an $\mathscr{M}_0$-subobject of S with $i_A = i_S i'_A\}$. $(S, i_S) \in R_Q(X)$ iff $(X - Q\,\mathrm{hull}(S, i_S), j_S) = (S, i_S)$. Since $R_Q(X)$ is closed under intersections, it follows that $Q = Q' = Q_{R_Q}$.

Corollary 2.6. *There exists a 1-1 correspondence between the class of hull operators on $\mathscr{C}$ and the class of hull subobject operators on $\mathscr{C}$.*

3. Strong factorization structures and hull operators

In this section we show that there is a 1-1 correspondence between the class of hull operators and the class of strong factorization structures. The principal result is summarized in Theorem 3.1. The proof is completed by showing that each hull operator induces a strong factorization structure, which in turn induces a hull subobject operator.

Theorem 3.1. *There exists a* 1-1 *correspondence between the following* 3 *families*:
1) *The class of hull operators on* $\mathscr{C}$;
2) *The class of hull subobject operators on* $\mathscr{C}$;
3) *The class of strong factorization structures on* $\mathscr{C}$.

Proof. We recall (Definition 1.6) that if $f: X\to Y$ is a morphism and $f=me$ is its $(\mathscr{E}_0, \mathscr{M}_0)$ factorization, then $(f(X), m)$ is the image of f.

Let Q be a hull operator on $\mathscr{C}$. Define $\mathscr{E}_Q$ and $\mathscr{M}_Q$ as follows:

$$\mathscr{E}_Q=\{f: X\to Y: (Y-Q\text{ hull}((f(X), m), j)=(Y, \mathrm{id}_Y)\};$$

$$\mathscr{M}_Q=\{f: X\to Y: f\in\mathscr{M}_0, (Y-Q\text{ hull}((f(X), m), j)=(f(X), m)\}.$$

Then $(\mathscr{E}_Q, \mathscr{M}_Q)$ is a strong factorization structure on $\mathscr{C}$. That $\mathscr{E}_Q\cap\mathscr{M}_Q$ contains all isomorphisms, $\mathscr{M}_Q\subseteq\mathscr{M}_0$, and $\mathscr{M}_Q$ is closed under composition is obvious.

To show that $\mathscr{E}_Q$ is closed under composition, let $f: X\to Y, g: Y\to Z\in\mathscr{E}_Q$. Consider the following diagram, where the inside square is a pullback:

$$\begin{tikzcd}
f(X) \arrow[dr, dashed, "h"] \arrow[drr, bend left, "m_3"] \arrow[ddr, "ie_4"'] & & \\
& E \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d, "g"] \arrow[dl, dashed, "d"'] \\
& Z\text{-}Q\text{ hull}(gf(X), m_1) \arrow[r, "k"'] & Z
\end{tikzcd}$$

Using the notation of Proposition 1.8, we have the following $(\mathscr{E}_0, \mathscr{M}_0)$ factorizations: $gf=m_1e_1$, $g=m_2e_2$, $f=m_3e_3$, and $gm_3=m_4e_4$. $i: gf(X)\to Z-Q\text{ hull}(gf(X), m_1)$ is the morphism defined in condition 1 of Definition 2.1. k denotes $j_{gf(X)}: Z-Q\text{ hull}(gf(X), m_1)\to Z$. $gm_3=m_4e_4$ and $m_4=ki$ imply that $gm_3=kie_4$. Thus h exists. $m_3=p_2h$, and so $h\in\mathscr{M}_0$. That p_2 is an isomorphism follows from Property 4 of Definition 2.1 and the fact that $f\in\mathscr{E}_Q$. Thus the diagonal d exists. Let $d=m_5e_5$, $m_5\in\mathscr{M}_0$, $e_5\in\mathscr{E}_0$. Then there exists $j\in\mathscr{M}_0$ with $j: Z-Q\text{ hull}(kd(Y), km_5)\to Z-Q\text{ hull}(gf(X), m_1)$. Since $kd=g$ and $g\in\mathscr{E}_Q$, $Z-Q\text{ hull}(kd(Y), km_5)=Z$, and thus $Z-Q\text{ hull}(gf(X), m_1)=Z$. Therefore $\mathscr{E}_Q$ is closed under composition.

For unique factorization of a morphism $f: X\to Y$, let $f=me$, $m\in\mathscr{M}_0$, $e\in\mathscr{E}_0$. $i: f(X)\to Y-Q\text{ hull}(f(X), m)$ is the morphism defined in condition 1 of Definition 2.1. If the $Y-Q$ hull of $(f(X), m)=(Y-Q\text{ hull}(f(X), m), m_1)$ and $e_1=ie$, then $f=m_1e_1$ is clearly an $(\mathscr{E}_Q, \mathscr{M}_Q)$ factorization of f. To show uniqueness, suppose that $f=m_2e_2$, $m_2\in\mathscr{M}_Q$, $e_2\in\mathscr{E}_Q$, where $e_2: X\to Z$, $m_2: Z\to Y$. Consider the following commutative square:

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[d, "e_2"'] & f(X) \arrow[d, "m"] \arrow[dl, dashed, "d"'] \\ Z \arrow[r, "m_2"'] & Y \end{tikzcd}$$

$m_2 \in \mathscr{M}_0$, $e \in \mathscr{E}_0$ imply that there exists d such that $m_2 d = m$. Thus $d \in \mathscr{M}_0$, and therefore there exists $j \in \mathscr{M}_0$ with j: $Y - Q$ hull$(f(X), m) \to Z$. Let $e_2 = m_0 e_0$, $m_0 \in \mathscr{M}_0$, $e_0 \in \mathscr{E}_0$, where e_0: $X \to e_2(X)$ and m_0: $e_2(X) \to Z$. Then $f = m_2 e_2 = m_2 m_0 e_0 = me$. Since $(\mathscr{E}_0, \mathscr{M}_0)$ is a factorization structure we have $m_2 m_0 \in \mathscr{M}_0$, $(f(X), m)$ and $(e_2(X), m_2 m_0)$ are isomorphic, and therefore $(Y - Q$ hull$(f(X), m), m_1)$ and (Z, m_2) are also isomorphic. Thus, $f = m_1 e_1$ is a unique $(\mathscr{E}_Q, \mathscr{M}_Q)$ factorization of f, and $(\mathscr{E}_Q, \mathscr{M}_Q)$ is a strong factorization structure on $\mathscr{C}$.

Suppose that $(\mathscr{E}, \mathscr{M})$ is a strong factorization structure on $\mathscr{C}$. Define a hull subobject operator $R_{\mathscr{M}}$ as follows: $R_{\mathscr{M}}(X) = \{(S, i_S)$: $i_S: S \to X \in \mathscr{M}\}$. $R_{\mathscr{M}}(X)$ is closed under intersections since $\mathscr{M}$ is. If

$$\begin{tikzcd} E \arrow[r, "p_2"] \arrow[d, "p_1"'] & X \arrow[d, "f"] \\ S \arrow[r, "i_S"'] & Y \end{tikzcd}$$

is a pullback with $i_S \in \mathscr{M}$, then $p_2 \in \mathscr{M}$, and so $(E, p_2) \in R_{\mathscr{M}}(X)$. Finally, suppose that $(Y, i_Y) \in R_{\mathscr{M}}(X)$. If $(S, i_S) \in R_{\mathscr{M}}(Y)$, then $i_Y i_S \in \mathscr{M}$, where $i_Y i_S$: $S \to X$. Thus $(S, i_Y i_S) \in R_{\mathscr{M}}(X)$. If, on the other hand, $(S, i_Y i_S) \in R_{\mathscr{M}}(X)$, then $i_Y i_S \in \mathscr{M}$, which implies $i_S \in \mathscr{M}$, and thus $(S, i_S) \in R_{\mathscr{M}}(Y)$. Therefore $R_{\mathscr{M}}$ is a hull subobject operator on $\mathscr{C}$.

We have already seen (Corollary 2.6) that there is 1-1 correspondence between the class of hull operators on $\mathscr{C}$ and the class of hull subobject operators on $\mathscr{C}$. That there is a 1-1 correspondence between the class of strong factorization structures on $\mathscr{C}$ and the class of hull subobject operators on $\mathscr{C}$ follows from the fact that both $R(X)$ and $\mathscr{M}$ are closed under intersections.

Remark. If $(\mathscr{E}, \mathscr{M})$ is a strong factorization structure on $\mathscr{C}$, then $Q_{R_{\mathscr{M}}}$ can be obtained as follows: $(X - Q_{R_{\mathscr{M}}}$ hull$(S, i_S), j_S) = (C, m)$, where $i_S = me$, $m \in \mathscr{M}$, $e \in \mathscr{E}$. This is the technique used by Nakagawa in [8] to obtain closure operators for $(\mathscr{E}, \mathscr{M})$ additive in *TOP*.

4. Modifications of $(\mathscr{E}, \mathscr{M})$

1) $\mathscr{M} \not\subseteq \mathscr{M}_0$.

If $\mathscr{M} \not\subseteq \mathscr{M}_0$, let $\mathscr{M}_1 = \mathscr{M} \cap \mathscr{M}_0$, and let $\mathscr{E}_1$ be such that $(\mathscr{E}_1, \mathscr{M}_1)$ is a strong factorization structure on $\mathscr{C}$. Then $(\mathscr{E}_1, \mathscr{M}_1)$, and thus $(\mathscr{E}, \mathscr{M})$, induces a hull operator on $\mathscr{C}$. This correspondence obviously is not 1-1.

2) $\mathscr{E} \subseteq Epi$.

If we wish to have a strong $(\mathscr{E}, \mathscr{M})$ factorization with $\mathscr{E} \subseteq Epi$, then Theorem 3.1 remains valid if we make the following restrictions on the class of hull operators and on the class of hull subobject operators:

a) If Q is a hull operator on $\mathscr{C}$, then the $X - Q$ hull of (S, i_S) is an $\mathscr{M}_0$-subobject of the X-affine hull of (S, i_S).

b) If R is a hull subobject operator on $\mathscr{C}$, then $R(X)$ contains all extremal mono-subobjects of X.

References

[1] S. Baron, Reflectors as compositions of epireflectors, Trans. Amer. Math. Soc. 136 (1969) 499–508.

[2] G. Baumslag, Some aspects of groups with unique roots, Acta Math. 104 (1960) 217–303.

[3] H. Herrlich, Topologische Reflexionen und Coreflexionen, Lecture Notes in Math. 78 (Springer, Berlin, 1968).

[4] H. Herrlich, Limit operators and topological coreflections, Trans. Amer. Math. Soc. 146 (1969) 203–210.

[5] H. Herrlich and G. Strecker, Category Theory, 2nd Edition, (Heldermann, Berlin, 1979) Sigma Series in Pure Mathematics 1.

[6] J.F. Kennison, Full reflective subcategories and generalized covering spaces, Ill. J. Math. 12 (1968) 353–365.

[7] H. Lord, Hull operators on a category of spaces of continuous functions on Hausdorff spaces, Studia Mathematica 55 (1976) 225–237.

[8] R. Nakagawa, Factorization structures and subcategories of the category of topological spaces, J. Australian Math. Soc. 21 (Series A) (1976) 144–154.

[9] L. Nel, Initially structured categories and cartesian closedness, Can. J. Math. 27 (1975) 1361–1377.

[10] P. Ribenboim, Torsion et localisation de groupes arbitraires, Lecture Notes in Math. 740 (Springer, Berlin, 1979) 444–455.